

Parametric Surfaces and *X(PLORE)*

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Objectives of Assignment

1. To learn how to use *X(PLORE)* to plot parametrically defined surfaces.
2. To learn how to parameterize ordinary functions of two variables.
3. To learn how to parameterize surfaces defined in cylindrical coordinates.
4. To learn how to parameterize surfaces defined in spherical coordinates.
5. To learn how to parameterize surfaces of revolution.

I. Introduction to Parametrically Defined Surfaces

To parameterize a surface you need to understand two very simple, yet extremely important, ideas. First, every point on a surface has an x -coordinate, a y -coordinate, and a z -coordinate. Second, the x -coordinate is controlled by u and v , the y -coordinate is controlled by u and v , and the z -coordinate is controlled by u and v .

When you are trying to parameterize a surface, always ask yourself, “How can I use u and v to describe x ?” Then ask, “How can I use u and v to describe y ?” Finally, ask yourself, “How can I use u and v to describe z ?” Essentially, you will have to find one function for each coordinate.

You will need to determine three functions $x(u, v)$, $y(u, v)$, and $z(u, v)$. Once you determine these, *X(PLORE)* will plot the surface for you.

II. Simple Examples Based on Functions of Two Variables

Suppose you want to parameterize the surface $f(x, y) = x^2 + y^2$. Each point of this surface has an x coordinate, a y coordinate, and a z coordinate. The z coordinate depends on both x and y since $z = x^2 + y^2$, but the x and y coordinates are independent variables.

Example 1. Find a parameterization for the surface given by $f(x, y) = x^2 + y^2$.

Since x and y are independent variables, just let $x = u$ and $y = v$. That is, let

$$\begin{aligned}x(u, v) &= u \\y(u, v) &= v.\end{aligned}$$

Now finding a parameterization of z is just a matter of using substitution. Simply put u in for x and v in for y :

$$\begin{aligned} z &= x^2 + y^2, \text{ or} \\ z &= u^2 + v^2, \text{ or} \\ z(u, v) &= u^2 + v^2. \end{aligned}$$




A complete parameterization of $f(x, y) = x^2 + y^2$ is given by:

$$\begin{aligned} x(u, v) &= u \\ y(u, v) &= v \\ z(u, v) &= u^2 + v^2. \end{aligned}$$

This is a very simple method of finding a parameterization for the graph of the function of two variables, or for any equation of three variables which can be solved for one of the variables.

To make X(PLORE) plot this in light green, simply type and enter the following five lines.

```
Color(lightgreen)
x(u, v)=u
y(u, v)=v
z(u, v)=u^2+v^2
Pa3(x(u, v), y(u, v), z(u, v), u = -2 to 2, v = -2 to 2)
```

X(PLORE) is probably better than *Mathematica* for exploring parameterized surfaces because it is easier to rotate the surfaces interactively. When the plotted surface is on the screen, just press , , or  to rotate it.

To plot more surfaces, just change the definition lines of x , y , and z , which have already been typed in. Don't retype all four of the important lines.

Example 2. Find a parameterization for the plane given by $3x - 2y + z = 6$.

To parameterize the plane $3x - 2y + z = 6$, first solve for z

$$z = 6 - 3x + 2y,$$

and then use substitution as shown above

$$\begin{aligned} x(u, v) &= u \\ y(u, v) &= v \\ z(u, v) &= 6 - 3x + 2y. \end{aligned}$$

In some cases, x and y might not be the independent variables. That's OK, just use u and v for whichever variables can be independent and keep x , y and z in the same order as before in the *Pa3* line.

Example 3. Find a parameterization of the surface given by $x^2 + y - z^2 - xz = 2$.

To parameterize $x^2 + y - z^2 - xz = 2$, solve for y instead of z . Then since $y = 2 - x^2 + z^2 + xz$, switch the roles of y and z when substituting:

$$\begin{aligned}x(u, v) &= u \\y(u, v) &= 2 - u^2 + v^2 + u * v \\z(u, v) &= v.\end{aligned}$$

Plot this for $-5 \leq x \leq 5$ and $-5 \leq z \leq 5$.

III. Parameterizing Surfaces Defined in Cylindrical Coordinates

Often you can simply let $r = u$, $\theta = v$, and then think of x as $r\cos(\theta)$ and y as $r\sin(\theta)$. A simple example of this is the cone defined by the equation $z = r$.

Example 1. Parameterize and plot the cone given by $z = r$.

First ask yourself, "What is the x coordinate, what is the y coordinate, and what is the z coordinate?" The answers are straightforward if you let $r = u$ and $\theta = v$:

$$\begin{aligned}x &= r\cos(\theta), \text{ so } x = u\cos(v); \\y &= r\sin(\theta), \text{ so } y = u\sin(v); \text{ and} \\z &= r, \text{ so } z = u.\end{aligned}$$

A parameterization of this cone is:

$$\begin{aligned}x(u, v) &= u\cos(v) \\y(u, v) &= u\sin(v) \\z(u, v) &= u\end{aligned}$$

Plot this with $-3 \leq z \leq 3$. (First determine what is u and what will its bounds have to be? Then determine what is v and what will its bounds have to be?)

Example 2. Find a parameterization of the cylinder given by $x^2 + y^2 = 9$.

For this surface, there are no restrictions on z , but r is always 3. This means z is independent and r is a constant. Since r is constant you must set $r = 3$. That frees up u . Since z is independent, set $z = u$. Therefore, a parameterization is:

$$x(u, v) = 3\cos(v), \quad y(u, v) = 3\sin(v), \quad z(u, v) = u.$$

IV. Parameterizing Surfaces Defined in Spherical Coordinates

This is somewhat similar to parameterizing surfaces defined with cylindrical coordinates, but now you have to think of x , y , and z in terms of \mathbf{r} , \mathbf{q} and \mathbf{j} . You also have to figure out how \mathbf{r} , \mathbf{q} and \mathbf{j} can be replaced, or substituted for, by u and v . Notice that you always end up writing three variables in terms of two: No matter what the original description is for the surface, in the end, it's always parameterized by simply u and v .

Example 1. Parameterize the bottom half of the sphere of radius 2 centered at the origin.

You will need to use the relationships:

$$\begin{aligned}x &= r\cos(\mathbf{q})\sin(\mathbf{j}), \\y &= r\sin(\mathbf{q})\sin(\mathbf{j}), \\z &= r\cos(\mathbf{j}).\end{aligned}$$

But how do they tie in with u and v , and how do you get half of a sphere?

For this sphere, $r=2$. \mathbf{q} and \mathbf{j} are variables and independent, so u and v should be used for \mathbf{q} and \mathbf{j} . Simple substitution then yields the following parameterization:

$$\begin{aligned}x(u, v) &= 2\cos(u)\sin(v), \\y(u, v) &= 2\sin(u)\sin(v), \\z(u, v) &= 2\cos(v).\end{aligned}$$

The bottom half of the sphere is where $\frac{\mathbf{p}}{2} \leq \mathbf{j} \leq \mathbf{p}$ and $0 \leq \mathbf{q} \leq 2\mathbf{p}$. Set the bounds on u and v accordingly.

V. Parameterizing Surfaces of Revolution

To parameterize these surfaces you will have to sketch and visualize the revolution, and apply a little bit of trigonometry. Two of the three variables x , y , and z will end up getting multiplied by trig functions. You will notice that the variable which is not multiplied by a trig function is the variable whose axis is the center of the revolution. Figuring out why this is so will probably help you construct accurate parameterizations.

Example 1. Find a parameterization of the surface generated by revolving the curve $y = x^2$, $0 \leq x \leq 2$, around the x -axis.

First sketch the curve in the x - y plane. Really. Sketch it out right now or you will get hopelessly lost.

As this curve is spinning around the x -axis, you once again have to ask yourself, “What is the x -coordinate of each point on the surface, what is the y -coordinate, and what is the z -coordinate?” To figure this out, start with one point on the curve and think of where it is after the curve has revolved through an angle of α . Looking at your drawing, you should see that the y -coordinate changes as the point revolves around the x -axis. Certainly the z -coordinate changes, too, because the curve moves out of the x - y plane as it revolves. However, for that individual point, the x -coordinate does not change during the revolution. (Hmmm . . . What’s the sense of having a revolution if things aren’t going to change?)

Since the x -coordinate does not depend on the angle of revolution, x is independent of α . The y -coordinate depends on x (because $y = x^2$), and on α . The z -coordinate acts a lot like the y -coordinate (which won’t make any sense if you didn’t draw a little sketch to follow along).

For a given point on the curve, both y and z depend on x and α . Applying a little trigonometry, you can see that after revolving through an angle of α , $y = x\cos(\alpha)$ and $z = x\sin(\alpha)$. Now substitute u and v for the independent variables x and α . The result is:

$$\begin{array}{ll} x = x & x(u, v) = u \\ y = x^2\cos(\alpha) & \text{or} \quad y(u, v) = u^2\cos(v) \\ z = x^2\sin(\alpha) & z(u, v) = u^2\sin(v) \end{array}$$

Try to think of the appropriate bounds for u and v and then plot the surface.

Although the previous example didn’t say anything about α in the statement of the problem, you have to introduce α because each point of the surface is a result of the curve revolved through an angle.

Example 2. Find a parameterization of the surface generated by revolving the curve $y = x^2$, $0 \leq x \leq 2$, around the y -axis.

Now the y -coordinate will be independent of α . The x -coordinate and z -coordinate will depend on α . Since y is independent of α , you might as well solve for x to make it the independent variable:

$$x = +\sqrt{y} \quad \text{with} \quad 0 \leq y \leq \sqrt{2}.$$

Then:

$$\begin{array}{ll} x = \sqrt{y} \cos \alpha, & \text{or} \quad x(u, v) = \sqrt{u} \cos v, \\ y = y, & y(u, v) = u, \\ z = \sqrt{y} \sin \alpha & z(u, v) = \sqrt{u} \sin(v). \end{array}$$

VI. Practice Problems

- Find a parameterization of the part of the plane $2x + y - 3z = 12$ which is in front of or behind the following part of y - z plane: $-2 \leq y \leq 3$ and $1 \leq z \leq 2$.
Hint: use y and z as the independent variables.
- Find a parameterization of the part of the surface $f(x, y) = x^2 + y^2 + 2$ which is above the right half of the circle $x^2 + y^2 = 4$.
Hint: with all these circular shapes, consider switching to cylindrical coordinates.
- Find a parameterization of the part of the surface $f(x, y) = x^2 + y^2 + 2$ which is above the circle $x^2 - x + y^2 = 0$.
Hint: switching to cylindrical coordinates will not work here because both u and v must be independent variables. Switching to cylindrical coordinates would force the bounds on u to be $0 \leq u \leq 2\cos(v)$ which makes u dependent on v . The only way of doing this problem is to first use a translation of the x -axis so the circle is centered at the origin. Then use cylindrical coordinates on the shifted axes. After this is done, envision the axes shifted back to their original position.
- Find a parameterization of the hyperboloid of one sheet given by the equation: $x^2 + 3y^2 - 2z^2 = 4$. Find the part which is within the region $-10 \leq x, y, z \leq 10$.
Hint: If you let $x = 0$, you will get a hyperbola in the y - z plane. Then just revolve one side of the hyperbola around the appropriate axis.

SOLUTIONS:

- $$x(u, v) = (12 - u + 3v)/2$$

$$y(u, v) = u$$

$$z(u, v) = v,$$

with $-2 \leq u \leq 3$ and $1 \leq v < 2$.
- $$x(u, v) = u\cos(v)$$

$$y(u, v) = u\sin(v)$$

$$z(u, v) = u^2 + 2,$$

with $0 \leq u \leq 2$ and $-\pi/2 \leq v < \pi/2$.
- First let $X = x - 1$, $Y = y$, and $Z = z$ to translate the axis.
Using substitution, the circle becomes $X^2 + Y^2 = 1$, and the surface becomes $f(X, Y) = (X + 1)^2 + Y^2$.

$$X(u, v) = u\cos(v)$$

$$Y(u, v) = u\sin(v)$$

$$Z(u, v) = (u\cos(v) + 1)^2 + (u\sin(v))^2 \text{ with } 0 \leq u \leq 3 \text{ and } 0 \leq v < 2\pi.$$
- After substituting $x = 0$ and solving for y , you get $y = \pm \frac{1}{3}\sqrt{4 + 2z^2}$.
Why solve for y instead of z ? Revolve the positive half around the z -axis. Why the z -axis?

$$x(u, v) = \frac{1}{3}\sqrt{4 + 2u^2} \sin(v)$$

$$y(u, v) = \frac{1}{3}\sqrt{4 + 2u^2} \cos(v)$$

$$z(u, v) = u,$$

with $-10 \leq u \leq 10$ and $0 \leq v \leq 2\pi$.