Integration by parts

The rule for differentiating a product of two functions of a single variable $x$ is

$$\frac{d}{dx} \left( f(x)g(x) \right) = f'(x)g(x) + f(x)g'(x).$$

A rule for differentiating a general product is called a Leibniz rule. We can use it to make an integration-by-parts formula, thus

$$f'(x)g(x) = \frac{d}{dx} \left( f(x)g(x) \right) - f(x)g'(x).$$

Therefore

$$\int_a^b f'(x)g(x) \, dx = f(x)g(x) \bigg|_a^b - \int_a^b f(x)g'(x) \, dx.$$ 

Similarly, there are Leibniz rules for other kinds of derivatives, such as div., grad. and curl, and other types of products, such as dot, cross and scalar multiply. Suppose $f$ and $g$ are any field quantities, maybe vector or scalar, and suppose $\star$ is any product that makes sense. Finally, let $\mathcal{D}$ be again any kind of derivative that makes sense. Then very roughly

$$\mathcal{D} (f \star g) = (\mathcal{D} f) \star g + f \star (\mathcal{D} g)$$

is the general form of a Leibniz rule.

Take for a first example the gradient. The Leibniz rule is

$$\nabla (fg) = (\nabla f) g + f (\nabla g),$$

so

$$\int_{\vec{r}_a}^{\vec{r}_b} f(\vec{r}) \nabla g(\vec{r}) \cdot d\vec{r} = f(\vec{r}_b)g(\vec{r}_b) - f(\vec{r}_a)g(\vec{r}_a) + \int_{\vec{r}_a}^{\vec{r}_b} g(\vec{r}) \nabla f(\vec{r}) \cdot d\vec{r}.$$ 

So this is the gradient’s integration by parts formula.

Now take the divergence. The fundamental theorem in this case is the divergence theorem,

$$\int_V \nabla \cdot \vec{v} \, d^3x = \int_{\partial V} \vec{v} \cdot \hat{n} \, dS,$$
where $V$ is some volume or 3D region, $\partial V$ is the closed surface pounding $V$, $dS$ is the surface area increment, with local outward normal $\hat{n}$, and $d^3x$ (same as $d\tau$ in Griffiths) is the volume element.

In this case the Leibniz rule involves two different kinds of functions, a scalar function $f(\vec{r})$, and a vector function $\vec{v}(\vec{r})$. Then $\star$ is just the ordinary product $f(\vec{r})\vec{v}(\vec{r})$. The vector identity
\[
\nabla \cdot (f(\vec{r}) \vec{v}(\vec{r})) = (\nabla f(\vec{r})) \cdot \vec{v}(\vec{r}) + f(\vec{r}) (\nabla \cdot \vec{v}(\vec{r})).
\]
works as a rather generalized Leibniz rule, and so we have an integration by parts formula
\[
\int_V (\nabla f(\vec{r})) \cdot \vec{v}(\vec{r}) \, d^3x = \int_{\partial V} f(\vec{r}) \vec{v}(\vec{r}) \cdot \hat{n} \, dS - \int_V f(\vec{r}) (\nabla \cdot \vec{v}(\vec{r})) \, d^3x,
\]
or, going the other way
\[
\int_V f(\vec{r}) (\nabla \cdot \vec{v}(\vec{r})) \, d^3x = \int_{\partial V} f(\vec{r}) \vec{v}(\vec{r}) \cdot \hat{n} \, dS - \int_V (\nabla f(\vec{r})) \cdot \vec{v}(\vec{r}) \, d^3x.
\]