Integration by parts

The rule for differentiating a product of two functions of a single variable x is

$$\frac{d}{dx}\Big(f(x)g(x)\Big) \;=\; f'(x)g(x) \;+\; f(x)\,g'(x).$$

A rule for differentiating a general product is called a Leibniz rule. We can use it to make an integration-by-parts formula, thus

$$f'(x)g(x) \;=\; \frac{d}{dx}\Big(f(x)g(x)\Big) \;-\; f(x)\,g'(x).$$

Therefore

$$\int_a^b f'(x)g(x)\,dx \;=\; f(x)g(x)\Big|_a^b \;-\; \int_a^b f(x)\,g'(x)\,dx.$$

Similarly, there are Leibniz rules for other kinds of derivatives, such as div. grad. and curl, and other types of products, such as dot, cross and scalar multiply. Suppose f and g are any field quantities, maybe vector or scalar, and suppose $\star$ is any product that makes sense. Finally, let $\mathcal{D}$ be again any kind of derivative that makes sense. Then very roughly

$$\mathcal{D}\,(f \star g) \;=\; (\mathcal{D}\,f) \star g \;+\; f \star (\mathcal{D}\,g)$$

is the general form of a Leibniz rule.

Take for a first example the gradient. The Leibniz rule is

$$\nabla(f\,g) \;=\; (\nabla f)\,g + \;f\,(\nabla g),$$

so

$$\int_{\vec{r}_a}^{\vec{r}_b} f(\vec{r})\nabla g(\vec{r})\cdot d\vec{r} \;=\; f(\vec{r}_b)g(\vec{r}_b) \;-\; f(\vec{r}_a)g(\vec{r}_a) \;+\; \int_{\vec{r}_a}^{\vec{r}_b} g(\vec{r})\nabla f(\vec{r})\cdot d\vec{r}.$$

So this is the gradient's integration by parts formula.

Now take the divergence. The fundamental theorem in this case is the divergence theorem,

$$\int_V \nabla\cdot\vec{v}\,d^3x \;=\; \int_{\partial V} \vec{v}\cdot\hat{n}\,dS,$$

where V is some volume or 3D region, ∂V is the closed surface pounding V, dS is the surface area increment, with local outward normal $\widehat{n}$, and d^3x (same as $d\tau$ in Griffiths) is the volume element.

In this case the Leibniz rule involves two different kinds of functions, a scalar function $f(\vec{r})$, and a vector function $\vec{v}(\vec{r})$. Then $\star$ is just the ordinary product $f(\vec{r})\vec{v}(\vec{r})$. The vector identity

$$\nabla \cdot (f(\vec{r})\,\vec{v}(\vec{r})) \;=\; (\nabla f(\vec{r})) \cdot \vec{v}(\vec{r}) \;+\; f(\vec{r})\;(\nabla \cdot \vec{v}(\vec{r}))\,.$$

works as a rather generalized Leibniz rule, and so we have an integration by parts formula

$$\int_V (\nabla f(\vec{r})) \cdot \vec{v}(\vec{r})\, d^3x \;=\; \int_{\partial V} f(\vec{r})\,\vec{v}(\vec{r}) \cdot \widehat{n}\, dS \;-\; \int_V f(\vec{r})\;(\nabla \cdot \vec{v}(\vec{r}))\; d^3x,$$

or, going the other way

$$\int_V f(\vec{r})\;(\nabla \cdot \vec{v}(\vec{r}))\; d^3x \;=\; \int_{\partial V} f(\vec{r})\,\vec{v}(\vec{r}) \cdot \widehat{n}\, dS \;-\; \int_V (\nabla f(\vec{r})) \cdot \vec{v}(\vec{r})\, d^3x.$$