Extensions of the Borsuk-Ulam Theorem

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Abstract

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One of the more well known results from topology is the Borsuk-Ulam Theorem. It states that any continuous function from an n-dimensional sphere to n-dimensional Euclidean space must map some pair of opposite points on the sphere to the same point in Euclidean space. This is often stated colloquially by saying that at any time, there must be opposite points on the earth with the same temperature and pressure. Unfortunately, most proofs of the theorem require advanced techniques learned from algebraic topology.

However, a simpler combinatorial lemma known as Tucker's Lemma can be shown to be equivalent to the Borsuk-Ulam Theorem. This gives the more appealing option of proving the Borsuk-Ulam Theorem by way of Tucker's Lemma.

For my thesis, I investigated this relationship between Tucker's Lemma and the Borsuk-Ulam theorem. I also investigated extensions of each to more complicated objects than spheres, such as a branched sphere, which is like a sphere but allows its coordinates to branch in more directions than the usual two found on a sphere.

I also investigated applications of these theorems. I examined how Tucker's Lemma and the Borsuk-Ulam theorem relate to fair division problems and how the extensions of each give more generalized fair division results.

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Chapter 1

Introduction

Topological theorems can be quite difficult to prove, as they often require many advanced techniques. Another approach, however, is to reduce the theorem to a combinatorial lemma that is often easier to prove.

1.1 The Brouwer Fixed Point Theorem

In 1910, Brouwer proved his famous fixed point theorem [2], which has become one of the most useful theorems in mathematics. A popular example of this theorem says that when you slosh coffee around in a cup, some point must return to its original place. Unfortunately, the proof is quite difficult in that it requires techniques from algebraic topology. In 1928, though, Sperner [13] discovered a simple combinatorial lemma involving triangulations of balls, which in 1929 was shown by Knaster, Kuratowski and Mazurkiewicz [7] to be equivalent to the Brouwer Fixed Point Theorem.

Therefore, one seeking to prove the Brouwer Fixed Point Theorem could now demonstrate Sperner's Lemma and then prove the equivalence, both of which are much simpler and more intuitive. Furthermore, when constructive versions of Sperner's Lemma emerged in the 1960's (e.g. see [3] and [8]), it was now possible to find the fixed points that were implied by Brouwer's Theorem. See [15] for a survey of such techniques.

1.2 The Borsuk-Ulam Theorem

Another theorem similar to Brouwer's is the Borsuk-Ulam Theorem [1] which says, colloquially, at any time there must be opposite points on the surface of the Earth with the same temperature and pressure. Again, although the standard proof involves the use of algebraic topology, one can prove the simpler combinatorial Tucker's Lemma [16] to which it is equivalent. In this thesis we will examine Tucker's Lemma, the Borsuk-Ulam Theorem, the relation between the two, and extensions of both.

In Chapter 2 we discuss Tucker's Lemma and various proofs that exist. In Chapter 3 we discuss the Borsuk-Ulam Theorem and its equivalence to Tucker's lemma. In Chapter 4 we discuss consequences of the Borsuk-Ulam Theorem, including such ideas as dividing a cake into two shares so that each of n people agree the division is fair, or dividing n different objects into two shares each.

In Chapter 5 we introduce the concept of a branched sphere and discuss the extension of Tucker's Lemma to a branched sphere. In Chapter 6 we show how Tucker's Lemma on a branched sphere leads to a version of the Borsuk-Ulam Theorem on a branched sphere. Finally, in Chapter 7 we discuss applications of the extended version of Tucker's Lemma, including dividing a cake into r shares so that each of n people agree the division is fair, or dividing n objects into r shares each.

Chapter 2

Tucker's Lemma

2.1 Some Definitions

Since Tucker's Lemma and the Borsuk-Ulam theorem both involve topological spheres, it will be helpful to define what we mean by these terms.

Definition 2.1.1 (n-dimensional Sphere) *Let S^n denote the n-*dimensional sphere $\{x \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} |x_i| = 1\}$*; this is homeomorphic to the unit n-dimensional Euclidean sphere* $\{x \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} |x_i|^2 = 1\}$.

Thus the sphere is the set of all points at unit distance (in some metric) from the origin. Note that a zero dimensional sphere is simply the set of two points $\{-1, 1\}$, unit distance from the origin in $\mathbb{R}^1$.

Definition 2.1.2 (n-dimensional Ball) *Let B^n denote the n-*dimensional ball $\{x \in \mathbb{R}^n | \sum_{i=1}^{n} |x_i| \leq 1\}$*; again, this is homeomorphic to the unit n-dimensional Euclidean ball* $\{x \in \mathbb{R}^n | \sum_{i=1}^{n} |x_i|^2 \leq 1\}$.

Thus the ball is the set of all points at unit distance or less (in some metric) from the origin.

A common way to view the sphere S^n is as the result of 2 copies of the ball B^n identified (e.g. "glued") along their boundaries, with the two copies being called *hemispheres*. In turn, the ball B^n may be obtained as the cone over the sphere S^{n-1}, in other words, introducing a new point x disjoint from S^{n-1} and "connecting" every point in S^{n-1} to x. Mathematically, B^n is obtained from $S^{n-1} \times [0, 1]$ with all points $S^{n-1} \times \{1\}$ identified to a single point x.

As Tucker's Lemma involves *triangulations* of the sphere into *simplices*, let us define these terms.

Definition 2.1.3 (Simplex) *An n-dimensional* simplex *is the smallest convex set containing $n+1$ affinely independent points.*

We will refer to the points forming the simplex as the *vertices* of that simplex. The most common way to view an n-simplex is as an n-dimensional triangle. These simplices have, as *faces*, other simplices that are obtained by deleting vertices from the simplex and examining the convex set that contains the remaining vertices. We will say that a vertex in one of these faces is *supported* by the vertices that are not deleted.

We will often say that two simplices σ and τ are opposite each other in a sphere or ball. This means that if the vertices of σ are $\{v_0, v_1, \ldots, v_n\}$ then the vertices of τ are $\{-v_0, -v_1, \ldots, -v_n\}$.

Definition 2.1.4 (Simplicial Complex) *A* simplicial complex T *is a finite collection of simplices such that*

1. *every face of a simplex is a simplex in the triangulation, and*
2. *any two simplices intersect in a face common to both*

If X is the set consisting of the union of the simplices in T, we sometimes say T is a *triangulation* of X.

Of particular interest will be the *octahedral subdivision* of a sphere S^n, which is the triangulation induced by the coordinate planes of R^{n+1}. In other words, the boundaries of the maximal simplices of the sphere occur along the planes of $\mathbb{R}^{n+1}$ where one or more coordinates are zero.

Definition 2.1.5 (Join) *Given two simplicial complexes T and T', their join $T \star T'$ is the complex that is formed by simplices whose vertices are taken from a simplex in T and a simplex in T'.*

The octahedral subdivision of a simplicial n-sphere is join of $n+1$ two-point sets: $S^n \cong \star_{i=1}^{n+1}\{i,-i\}$, as in Figure 2.1. In this case, one may look at a vertex of the join as the point of the sphere lying along a coordinate axis, and a simplex of the join as the orthant spanned by those coordinate axes corresponding to their respective vertices.

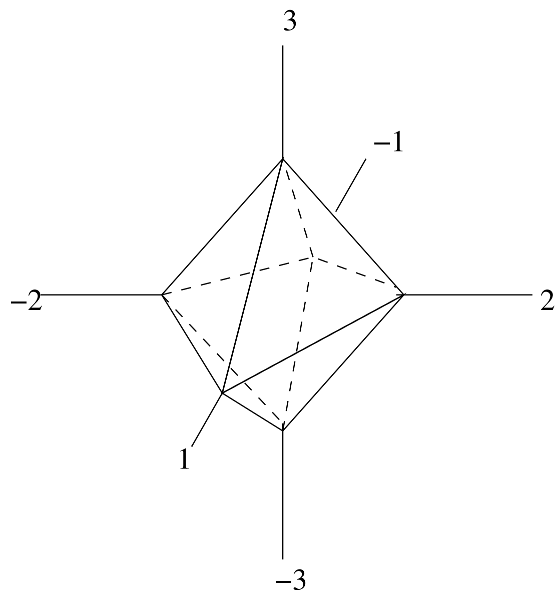

Figure 2.1: A two dimensional sphere is the join of three two-point sets.

Definition 2.1.6 (Adjacency) *Two distinct vertices are* adjacent *if they are contained within some 1-simplex of a triangulation.*

Definition 2.1.7 (Symmetric Triangulation) *A triangulation T of a sphere S^n is* symmetric *if for every simplex σ of T, the opposite simplex $-\sigma$ is also in T.*

Definition 2.1.8 (Sign) *Given a real or complex number z, define the* sign $sgn(z) = z/|z|$, *with the additional condition that $sgn(0) = 0$. Define the* sign vector *of a real or complex vector by taking the sign of each component, i.e.* $sgn([v_1, v_2, \ldots, v_n]^\top) = [sgn(v_1), sgn(v_2), \ldots, sgn(v_n)]^\top$.

2.2 Tucker's Lemma

In 1945, Tucker was looking for a way to use Sperner's Lemma to prove the Borsuk-Ulam Theorem [12]. Unfortunately, he did not find such a way. However, Tucker proved a similar lemma that quickly implies the Borsuk-Ulam theorem [16].

Theorem 1 (Tucker's Lemma) *Given a symmetric triangulation T of the octahedral division of S^n with vertex set V, any labelling $\lambda : V \to \{\pm 1, \ldots, \pm n\}$ such that $\lambda(-v) = -\lambda(v)$ for all $v \in V$ must assign some adjacent pair of vertices with opposite labels.*

Because Tucker's original proof was a proof by contradiction, his proof of the Borsuk-Ulam theorem was not constructive. Several years passed before Freund and Todd [6] found a constructive proof of Tucker's Lemma.

2.3 Freund & Todd's Proof of Tucker's Lemma

In 1981, Freund & Todd gave a constructive proof of a version of Tucker's lemma for the n-ball B^n. Because S^n can be created by taking two copies of B^n and identifying their boundaries, Tucker's lemma can be rephrased on B^n by only requiring that the boundary of B^n be oppositely labelled — the equivalent version of Tucker's lemma on S^n would be obtained by duplicating B^n, negating the labels, and then identifying the boundaries.

Theorem 2.3.1 (Tucker's Lemma on B^n) *Given a triangulation T of the octahedral subdivision of B^n with vertex set V that is symmetric along the boundary, any labelling $\lambda : V \to \{\pm 1, \ldots, \pm n\}$ such that $\lambda(-v) = -\lambda(v)$ for all v in the boundary must assign some adjacent pair of vertices with opposite labels.*

Our first proof will be that given by Freund & Todd [6].

Proof: Given a n-vector s of signs, we will say that a simplex $\sigma \in T$ is s-labelled if whenever s_i is nonzero, $s_i \cdot i$ is a label of some vertex of σ, so that the signs with the vector correspond to the signs of the vertices of σ. For any simplex $\sigma \in T$, the interior of σ is constrained by the coordinate planes and must therefore have the same sign throughout; therefore, we denote by $\text{sgn}(\sigma)$ this sign vector. Finally, if σ is $\text{sgn}(\sigma)$ labelled, we will say it is *completely labelled*.

We remark that if a simplex is completely labelled and lies in the boundary of the ball, then its antipodal simplex must also be completely labelled. Also, the zero simplex found at the origin is always completely labelled, as its sign vector is 0.

We now construct a graph G whose nodes are completely labelled simplices in T, saying that two simplices σ and τ are adjacent in G if $\sigma \cup \tau$ is $\text{sgn}(\sigma \cap \tau)$ labelled or if they are opposite each other in the triangulation of the boundary. Thus, any completely labelled simplex has either degree one or two in G, with a simplex having degree one if and only if it is the origin or a completely labelled simplex containing oppositely labelled adjacent vertices. Thus the graph G is a collection of disjoint cycles and paths. Hence, there must be an even number of simplices in G of degree one, and since one of these is the origin, there must be an odd number of completely labelled simplices that contain oppositely labelled adjacent vertices. □

As an example of the “path-following” proof Freund & Todd discovered, consider Tucker's Lemma on B^2, as shown in Figure 2.2.

We remark that the proof still works as long as there are an even number of completely labelled simplices in the boundary of the ball; the proof is constructive as long as the simplices can be paired together in some meaningful manner. This is done in the Freund & Todd proof by pairing opposite simplices together, but this does not need to be the case.

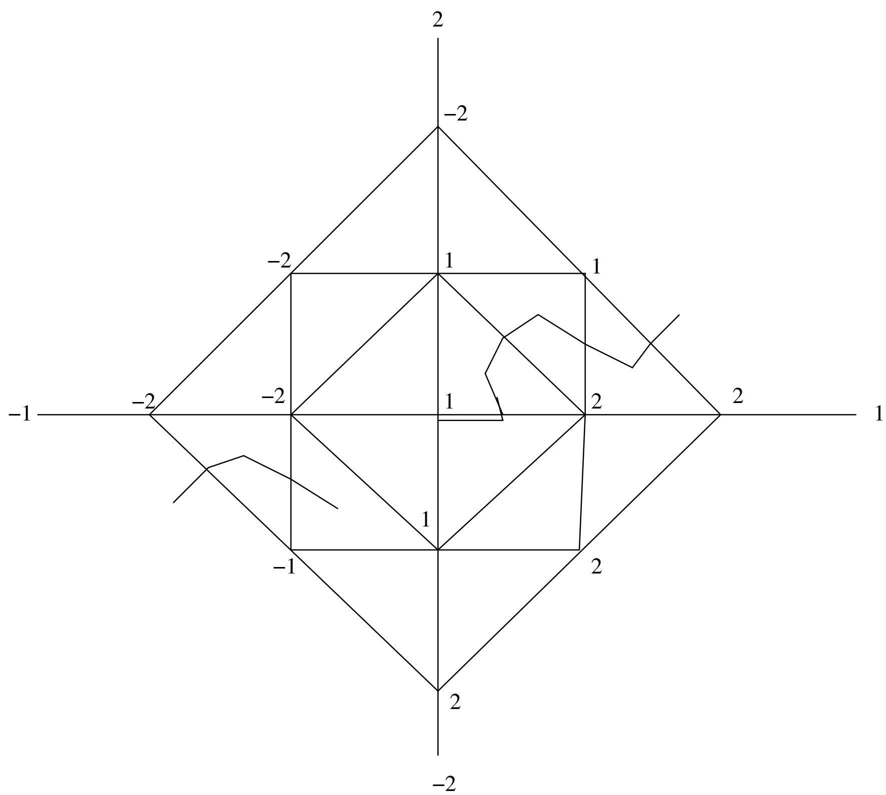

Figure 2.2: The path following argument given by Freund & Todd.

2.4 Fan's Proof of Tucker's Lemma

In 1952, Ky Fan proved a more generalized version of Tucker's lemma that did not require any relation between the labels and the dimension of the sphere [5]. Although Fan proved his theorem non-constructively, he used a pairing method to isolate the desired simplices. This method quickly lends itself to a constructive proof that we will present here. The theorem relies on counting the parity of *alternating* simplices.

Definition 2.4.1 (Alternating) *A simplex σ will be said to be* alternating *if its labels alternate signs when arranged according to increasing absolute value.*

We will also denote by $\beta(j_1, \ldots, j_{n+1})$ the number of n-simplices in the triangulation whose $n+1$ vertices receive the labels $j_1, \ldots, j_{n+1}$.

Theorem 2.4.2 (Fan's Extension of Tucker's Lemma) *Let T be a triangulation of the octahedral subdivision of S^n, with vertex set V. Consider a labelling $\lambda : V \to \{\pm 1, \ldots, \pm m\}$ such that $\lambda(-v) = -\lambda(v)$ for all $v \in V$ and no two adjacent vertices have opposite labels. Then:*

$$\sum_{1 \le k_1 < \cdots < k_{n+1} \le m} \beta(k_1, -k_2, \ldots, (-1)^n k_{n+1}) \equiv 1 \pmod 2$$

In other words, there are an odd number of full dimensional alternating simplices whose first label is positive. Note that for the sum to be non-void, we must have that $m \ge n+1$, which is the contrapositive of Tucker's Lemma.

Fan's proof for his theorem involved inducting on the dimension n and was not constructive. However, because he paired up related simplices, we may trace a path through those simplices to find the full dimensional alternating simplices.

Proof: Because the sphere has the octahedral subdivision, it can be oriented with the coordinate axis of $\mathbb{R}^{n+1}$. This allows us to define a *flag* $\mathcal{H}$ of hemispheres: $\mathcal{H} = H_0 \subset \cdots \subset H_n$, where $H_i = \{x \in S^n | x_i \ge 0, x_j = 0, i+1 \le j \le n+1\}$. Note that H_0 is a single point (where the x_1 axis intersects the sphere) and that H_n is half of the full sphere. We will denote by H_i' the anti-hemisphere that is opposite H_i, that is, $H_i' = \{x \in S^n | x_i \le 0, x_j = 0, i+1 \le j \le n+1\}$.

We define a graph G whose nodes are alternating $d-1$ and d-simplices carried by H_d or H_d', with the exception of the possible $(d+1)^{st}$ vertex, which may be freely labeled. We will say that two simplices σ and τ are adjacent in G if one is a facet of the other, at least one of σ and τ have their smallest label being positive, and $\sigma \cup \tau$ is in the hemisphere H_d. We will also say that σ and τ are adjacent if they are opposite each other in S^n, and the one with the negative smallest label is located in a flagged hemisphere H_d.

We may now note that every node in the graph has degree two (by construction) with the exception of the point as H_0 and any full dimensional alternating simplex with positive smallest label.

Since every graph must have an even number of nodes with odd degree, there must be an odd number of full dimensional alternating simplices with positive first element. □

In Figure 2.3, we have shown Fan's path following proof for the same ball as before. As with the proof by Freund & Todd, we may only examine the ball instead of the sphere by reflecting the triangulation to the other side of the ball and proceeding as if we were on the opposite side of the sphere whenever the triangulation must pass through the boundary of the ball.

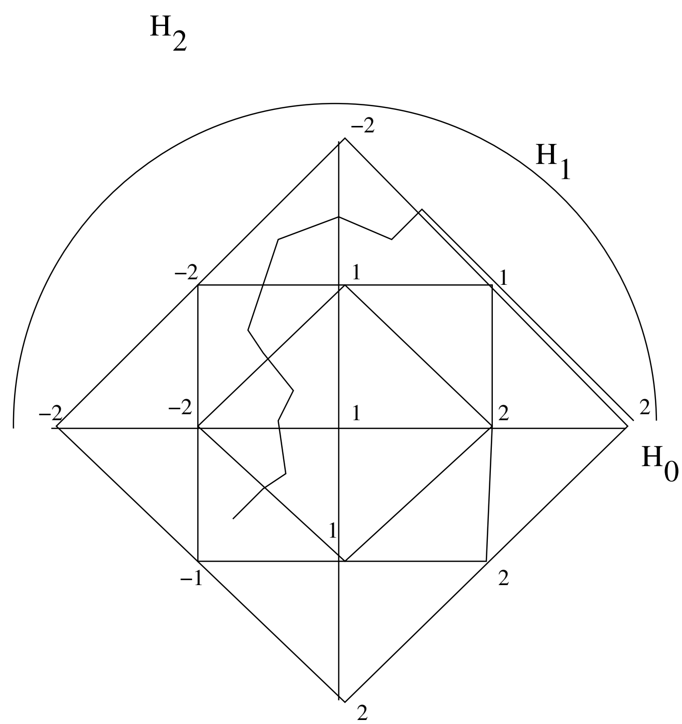

Figure 2.3: The path following argument derived from Fan.

In the event that $m = n + 1$, then there is only one possibility that the sum can use, a simplex whose labels are $\{1, -2, \ldots, (-1)^n(n+1)\}$. Furthermore, in this situation each label and its negative are interchangeable: switching the two labels in the sphere does not change any assumption of the theorem. Finally, we may examine a ball using this theorem by looking for simplices whose labels would be included in the sum if the other half of the sphere were included. Therefore, we have a simplified version of Fan's theorem.

Corollary 2.4.3 *Let T be a triangulation of the octahedral subdivision of S^n, with vertex set V. Given any labelling $\lambda : V \to \{\pm 1, \dots, \pm(n+1)\}$ such that $\lambda(-v) = -\lambda(v)$ for all $v \in V$ and no two adjacent vertices have opposite labels,*

$$\beta(1, 2, \dots, n+1) + \beta(-1, -2, \dots, -(n+1)) \equiv 1 \pmod 2$$

Either version of Fan's proof may then be used to prove Tucker's lemma on the sphere. When following the path described above, the path must start at H_0 and continue until reaching a simplex that has adjacent oppositely labeled vertices.

Chapter 3

The Borsuk-Ulam Theorem

In 1933, Borsuk proved his famous theorem, which asserts, for continuous functions $f : S^n \to \mathbb{R}^n$, the existence of antipodal points that get mapped to the same value. This has a similar flavor as the Brouwer fixed point theorem.

3.1 The Borsuk-Ulam Theorem

Theorem 2 *Given any continuous function $f : S^n \to \mathbb{R}^n$, there is some $x \in S^n$ such that $f(x) = f(-x)$.*

Colloquially, this is often stated by saying that at any time, there must be some point on the surface of the earth with the same temperature and pressure (assuming that these are continuous functions on the sphere).

3.2 Equivalent Statements

There are many statements that are equivalent to this theorem:

Theorem 3.2.1 *Given any continuous function $g : S^n \to \mathbb{R}^n$ such that $g(-x) = -g(x)$, there is some $x \in S^n$ such that $g(x) = 0$.*

The Borsuk-Ulam theorem easily implies this statement, as the only way for $g(x) = g(-x)$ is if both are zero. On the other hand, given a continuous $f : S^n \to \mathbb{R}^n$ that does not obey the antipodal behavior, we can define a function $g(x) = f(x) - f(-x)$. Then g obeys the antipodal property and thus has a zero, which corresponds to points in f being equal. We will be using this equivalent version throughout the paper.

Theorem 3.2.2 *There is no continuous retraction of S^n to S^{n-1} that is antipode preserving (i.e., antipodes get mapped to antipodes).*

If we had such a retract, then it would also be a continuous function from Theorem 3.2.1 (by including S^{n-1} into $\mathbb{R}^n$) that does not have a zero. On the other hand, if we had a function from Theorem 3.2.1 that did not have a zero, we could map the range space $\mathbb{R}^n$ onto S^{n-1} by normalizing every point in $\mathbb{R}^n$. This would be continuous because 0 would not be in the range space, and this would also give us the retraction required by Theorem 3.2.2.

3.3 Tucker's Lemma and the Borsuk-Ulam Theorem

Because of the similarity between the Brouwer fixed point theorem and the Borsuk-Ulam theorem, Lefschetz hoped that either Brouwer's or Sperner's theorems would be able to prove the Borsuk-Ulam theorem in a fairly simple way [12]. Unfortunately, such a proof remained undiscovered.

Later, A. W. Tucker proved his combinatorial lemma which allowed for a much simpler proof of the Borsuk-Ulam theorem [16]. However, his proof was by contradiction [6], which unfortunately did not lead immediately to a method to find the points given by the Borsuk-Ulam theorem.

Later, theorems such as that of Freund & Todd [6] and of Fan [5] were proved constructively, which now allows one to find the points given by the Borsuk-Ulam Theorem 3.2.1.

We now show the equivalence of Tucker's Lemma and the Borsuk-Ulam Theorem.

Theorem 3.3.1 *Tucker's Lemma implies the Borsuk-Ulam Theorem.*

Proof: Given a continuous function $f : S^n \to \mathbb{R}^n$ such that $f(-x) = -f(x)$, we shall find a point x_0 such that $f(x_0) = 0$. In light of Theorem 3.2.1, this will establish the Borsuk-Ulam Theorem. We will do this by finding a sequence of points $\{x_i\}_{i=1}^{\infty}$

such that $|f(x_i)| < 1/i$ (under the supremum norm) and then examine the sequence as $i \to \infty$.

Since f is a continuous function on a compact domain, it is uniformly continuous. Therefore, given i, triangulate S^n so that any two adjacent vertices have images under f that are within $1/i$ of each other. For each vertex v of the triangulation, let $j(v) = \text{argmax}_k |f_k(v)|$, and set $\lambda(v) = j(v)\text{sgn} f_{j(v)}(v)$. This induces a Tucker labelling of the sphere, as opposite must have opposite images under f, and therefore opposite labels under λ.

By Tucker's Lemma, therefore, there must be some adjacent pair of vertices that are oppositely labelled. Without loss of generality, assume that these labels are 1 and -1. Because a vertex was labelled 1, we know that its largest component must be the first one, and it must be positive. Similarly, we know that the largest component of the negative vertex must be the first one, and it must be negative. Because these vertices are adjacent, they must have images that are within $1/i$ of each other. Specifically, the first coordinates of their images must be within $1/i$ of each other. Because they are within $1/i$ of each other, with one positive and one negative, they must be within $1/i$ of 0. Since these were the largest coordinates, the entire vector must be within $1/i$ of 0.

Therefore, we have found a sequence of vectors $\{x_i\}$ where the image of each vector is within $1/i$ of the origin. Because we have a sequence on a compact set, there must be a convergent subsequence, and this subsequence must converge to a limit point whose image is zero. □

Theorem 3.3.2 *The Borsuk-Ulam Theorem implies Tucker's lemma.*

Proof: Given a triangulated n-sphere with vertex set V and labelling function λ, for all vertices $v \in V$, define $f(v)$ to be the unit vector along the $\lambda(v)^{th}$ axis, in the positive direction if $\lambda(v) > 0$ and the negative direction if $\lambda(v) < 0$. We now extend f linearly to the rest of the sphere by mapping any simplex of the triangulation to the simplex spanned by the images of its vertices. Therefore we have created a continuous $f : S^n \to \mathbb{R}^n$.

Since opposite vertices must have had opposite labels and the linear extension for a point and its antipode must be equivalent (i.e. opposite points on the sphere are the same combinations of the opposite vertices that span that point's simplex), opposite vertices of the triangulation must have opposite values under f. Since opposite points on the sphere that are not vertices are the same linear extensions of opposite points in the sphere, they must have opposite values under f. Therefore, by the Borsuk-Ulam theorem, there must be some point in the sphere that is mapped to 0. However, the only way for a simplex to cover the origin under f is for the simplex to contain oppositely labelled vertices, which gives the conclusion of Tucker's Lemma. □

In light of the above, we may conclude:

Corollary 3.3.3 *The Borsuk-Ulam Theorem and Tucker's Lemma are equivalent.*

Chapter 4

Fair Division

The Borsuk-Ulam theorem and Sperner's Lemma can be used to prove many interesting results. We will start by explaining how they can be used to divide a cake in two portions so that each of n people agree that the cake is evenly halved [11].

4.1 Dividing a Cake

Suppose we had a cake that we wish to divide into two equal portions. The equality of the shares will be assessed by a group of n people, who may have different ideas about what would be a fair division of the cake. We seek a division into shares so that all n people assess the two portions to be equal. We will make a series of parallel cuts along the cake and assignments of the pieces that can be parametrized by the points on the sphere. We will then find a division of the cake into two pieces using $n-1$ cuts that will satisfy everyone within a preset amount ϵ.

This can be formalized as follows: Given a bounded set in $\mathbb{R}^k$ and a n different measures on the set, we want to find a series of cuts by $(k-1)$-dimensional parallel hyperplanes of the set into two portions such that each measure agrees that the portions are equal. We will require that the measures be continuous (i.e. no point masses) and bounded.

We start by moving the cake so that it is tangent to the hyperplane $x_1 = 0$ and scaling it so that it is also tangent to the hyperplane $x_1 = 1$; we will be using cuts by hyperplanes $x_1 = c$, where $0 < c < 1$. Now any point in the n-sphere S^n can be associated with a division by n planes and an assignment of the $n+1$ pieces to portions in the following way: let $|z_i|$ = width of piece i and let $\text{sgn}(z_i) = \pm 1$ according as the i^{th} pieces is assigned to the first or second portion. Notice that $\sum_{i=1}^{n+1} |z_i| = 1$, as required by definition 2.1.1, the n-dimensional sphere.

We now define a function $f : S^n \to \mathbb{R}^n$ that takes a point on the sphere (that is, a set of cuts) and makes each coordinate that person's relative difference in measure between the two pieces. Since antipodal points on the sphere correspond to switching the pieces, each person will measure the same relative difference, but will choose the other piece as their favorite. Therefore, antipodal points will have opposite images, and f satisfies the criterion of the Borsuk-Ulam theorem.

By the Borsuk-Ulam theorem, there must be a zero of f. This corresponds to everyone agreeing that the cake has been evenly divided, as we desired. This zero can be found to within any desired accuracy, as was shown in the previous chapter.

Finally, note that the $n-1$ cuts are the minimum that will be able to fairly divide the cake. If everyone were to prefer a disjoint pieces of the cake (which would be equivalent to the measures being mutually singular), then one would need $n-1$ cuts to fairly divide the cake.

4.2 Ham Sandwich Theorem

Another interesting application of the Borsuk-Ulam Theorem and Tucker's Lemma is known as the Ham Sandwich Theorem. We will first state and prove a similar theorem.

Theorem 4.2.1 *Let μ be a continuous measure on $\mathbb{R}^n$. Given $n-1$ measurable sets and an arbitrary point x_0, there exists some $(n-1)$-hyperplane through x_0 which bisects each of the sets.*

Proof: The set of oriented hyperplanes through x_0 is homeomorphic to the sphere S^{n-1}, by identifying the normal vector of a plane with the corresponding point on the sphere. For any point on the sphere, we define a function which measures the difference of each set on either side of the plane (with the side toward the normal being arbitrarily positive). Since negating the normal vector would give the same plane but switch the bisected sets, the measured differences would negate as well. Therefore, by the Borsuk-Ulam Theorem 3.2.1, there would be a simultaneous zero, meaning that each set has been bisected. □

This now brings us to the Ham Sandwich Theorem.

Theorem 4.2.2 (The Ham Sandwich Theorem) *Given n Lebesgue measurable sets in $\mathbb{R}^n$, there exists some $(n-1)$-hyperplane which bisects each of the sets.*

Therefore, given a portion of ham, cheese, and bread in three dimensions, you could use one cut of a knife to split each in half and allow you to enjoy a sandwich with a friend.

Proof: Given the n sets, embed each of them in $\mathbb{R}^{n+1}$ where $x_{n+1} = 1$. By Theorem 4.2.1, there exists some n-hyperplane through the origin of $\mathbb{R}^{n+1}$ which bisects each of the sets under n-dimensional Lebesgue measure. □

Chapter 5

Branched Spheres

5.1 Some Definitions

My research focused on an object that we call a *branched sphere*, which is similar to a sphere except that coordinates may receive signs other than simply positive and negative. To help explain what this is, we begin by defining a few helpful ideas:

Definition 5.1.1 (ω, $\mathbb{Z}_r$, and r-axes) *Given a number $r \in \mathbb{Z}^+$, let $\omega \equiv \exp(2\pi i/r)$. The variable r will not usually be stated, but will be understood in context. We also define $\mathbb{Z}_r$ to be the cyclic group of order r, represented by the r^{th} roots of unity, $\{\omega^j\}$. Finally, we will define the r-axes of $\mathbb{C}$ to be the set $\{z \in \mathbb{C} | sgn(z) \in \mathbb{Z}_r \cup \{0\}\}$.*

The requirement that $\text{sgn}(z) \in \mathbb{Z}_r \cup \{0\}$ forces z to lie in a certain portion of the complex plane, as shown in Figure 5.1.

Definition 5.1.2 (Branched Sphere) *Let $S^{n,r}$ denote the* branched sphere $\{z \in \mathbb{C}^{n+1} | \sum_{j=1}^{n+1} |z_j| = 1, sgn(z_j) \in \mathbb{Z}_r \cup \{0\}\}$.

As with the usual sphere, the branched sphere is the set of all points at unit distance (in some metric) from the origin that lie along the r-axis of $\mathbb{C}^{n+1}$. Note also that a zero dimensional r-branched sphere is simply the set of r points $\mathbb{Z}_r$, unit distance from the origin in $\mathbb{C}^1$. The concept of a branched sphere was first discussed by Simmons and Su [11]. Analogous to the Euclidean sphere, we may also define a branched ball.

Definition 5.1.3 (Branched Ball) *Let $B^{n,r}$ denote the* branched ball $\{z \in \mathbb{C}^n | \sum_{j=1}^{n} |z_j| \leq 1, sgn(z_j) \in \mathbb{Z}_r \cup \{0\}\}$.

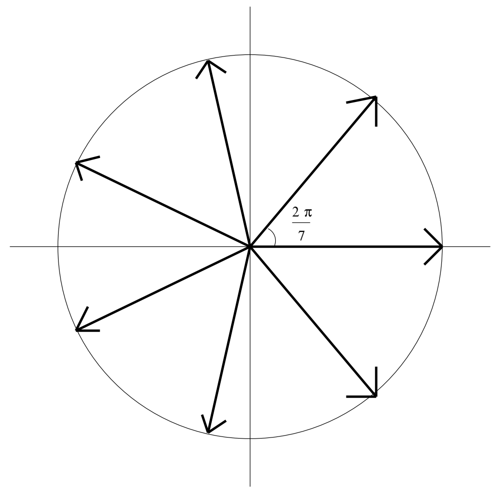

Figure 5.1: The requirement that $\text{sgn}(z) \in \mathbb{Z}_r \cup \{0\}$ causes z to lie in a certain portion of the complex plane.

Thus, the ball is the set of all points at unit distance or less (in some metric) from the origin that lie along the r-axis of $\mathbb{C}^n$.

We will allow the group $\mathbb{Z}_r$ to act on the branched sphere and ball by setting $\omega^i \circ z_j = \omega^i z_j$ and having ω^i act on z component-wise. Therefore, an element of $\mathbb{Z}_r$ acts on z by cycling the signs of its components.

It is important to note that if $r = 2$, then $\omega = -1$ and the branched sphere and ball are equivalent to the our previous definition (see definitions 2.1.1 and 2.1.2). In this case, the action described above gives rise to the antipodal map on the sphere and ball.

Similar to the usual sphere and ball, there are a few helpful ways to think about the branched sphere and ball. The branched sphere $S^{n,r}$ is obtained from r copies of the branched ball $B^{n,r}$ by identifying (e.g. gluing) them along their respective boundaries, with the r copies being called *polyspheres*. The branched ball $B^{n,r}$ is the result of taking the cone over the branched sphere $S^{n-1,r}$. As noted before, the zero dimensional branched sphere $S^{0,r}$ is the set of r points, often realized as the r^{th} roots of unity.

By analogy from the normal sphere and ball, we make several definitions.

Definition 5.1.4 (Symmetric Triangulation) *A triangulation T of $S^{n,r}$ is* symmetric *if for any $\sigma \in T$, we have that $\omega^i \circ \sigma \in T$ for all i.*

Definition 5.1.5 (Octahedral Subdivision) *The* octahedral subdivision *of the branched sphere and ball is that induced by the r-axis of complex space.*

As in the case of the simplicial sphere, the octahedral subdivision of a simplicial r-branched n-sphere may be viewed as the join of $(n+1)$ r-element sets: $S^{n,r} \cong \star_{i=1}^{n+1}\{\{i\} \times \mathbb{Z}_r\}$. Just as for the usual sphere, a vertex of the join may be viewed as the point of the branched sphere that lies along a coordinate r-axis, while a simplex of the join may be viewed as the orthant spanned by the corresponding coordinate r-axis.

Any realization of the branched sphere is difficult to visualize. However, we attempt to do so here. Because the r-branched 0-sphere is a set of r discrete points, the r-branched 1-ball is a collection of line segments with a common endpoint, as in Figure 5.2. In Figure 5.3 we see the 3-branched 2-ball. Because the ball is not planar, it must be broken into two separate pieces to be conveniently viewed. However, the triangles spanned by $(1,0)$, $(0,1)$, and $(0,0)$ are equivalent, as are those spanned by $(\omega,0)$, $(0,\omega)$, and $(0,0)$ and those spanned by $(\omega^2,0)$, $(0,\omega^2)$, and $(0,0)$. They simply appear twice for ease of viewing.

In labelling the vertices of a triangulation, each vertex will receive a sign from $\mathbb{Z}_r$ and a value from $\mathbb{N}$. This is also analogous to the $r=2$ case, where the sign was ± 1.

Definition 5.1.6 (Fully Signed) *A labelled simplex is* fully signed *if it has r vertices that have the same value but different signs.*

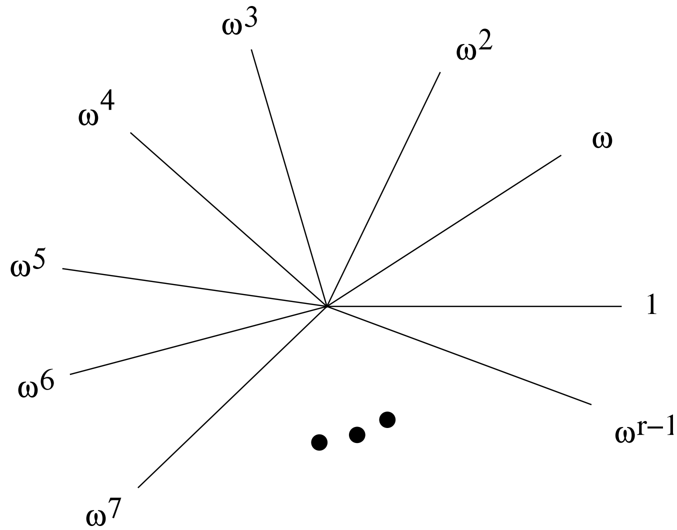

Figure 5.2: The r-branched 1 ball is a set of line segments with a common endpoint.

5.2 Tucker's Lemma on a Branched Sphere

We are now ready to state Tucker's lemma on a branched sphere. Unfortunately, I was unable to find a combinatorial proof of the lemma, although Ziegler seems to have proven it for prime values of r[17]. Therefore, everything in this chapter is a conjecture, and everything in the following two chapters are conjectures that depend on the existence of the following theorem.

Theorem 3 (The $\mathbb{Z}_r$-Tucker Lemma) *Given a symmetric triangulation of the octahedral subdivision of the branched sphere $S^{n(r-1),r}$ with vertex set V and even r, then any labeling $\lambda : V \to \mathbb{Z}_r \times \{1, 2, \ldots, n\}$ that is equivariant under the $\mathbb{Z}_r$ action (i.e. $\lambda_1(\omega^j v) = \omega^j \lambda_1(v)$) must make some simplex fully signed.*

In the case $r = 2$, we have already noted that the branched sphere reduces to the normal sphere; we can also see that $\mathbb{Z}_r = \{1, -1\}$. Therefore, we have a function $\lambda : V \to \{\pm 1, \pm 2, \ldots, \pm n\}$ that gives some simplex the same value but different signs, as in Tucker's original lemma. This Lemma is similar to that presented by

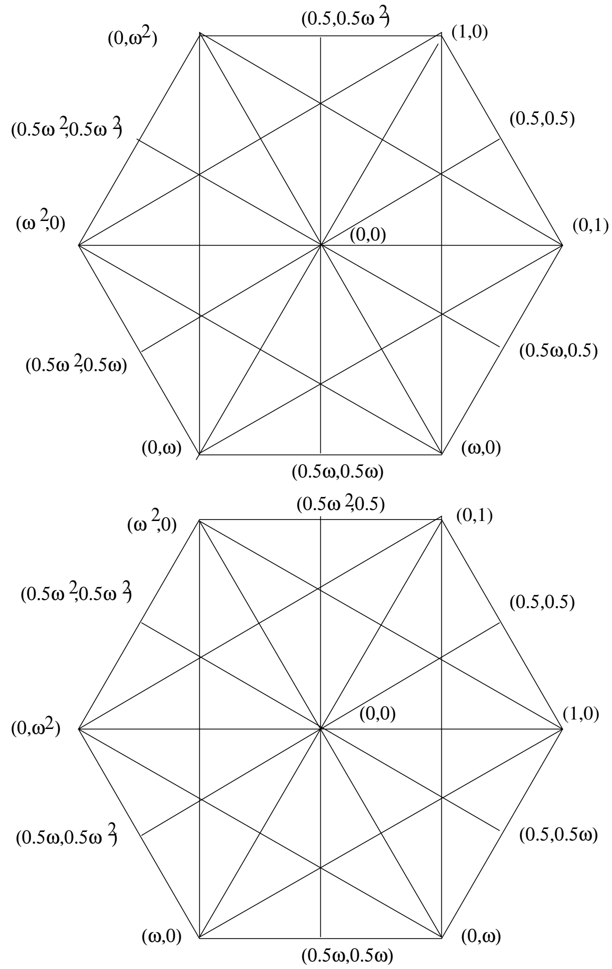

Figure 5.3: The first barycentric subdivision of the 3-branched 2-ball, which is broken into two pieces for ease of viewing.

Simmons and Su [11], although they require equivariance under the S_n action, and that presented by Ziegler [17], although he requires r to be prime instead of even. Although we shall require r to be even for the proof of the theorem, there does not seem to be any inherent reason for the theorem to require r to be even — it therefore seems likely that the theorem is true for all r.

Unfortunately, a proof similar to that by Freund & Todd does not seem to work in this case, as the additional dimensions of the branched sphere make it difficult to define what orthant a completely labelled simplex should be in. In light of this, we would like to prove a theorem that is related to Fan's version of Tucker's lemma.

To do so, we would first define a set K of allowable labelings and denote by K_n those labelings of length $n+1$, which would label an n-simplex. Similar to Fan's theorem, we would define $\beta(\tau, k)$ to be the indicator function that would return 1 if τ has k as its labeling and 0 otherwise. Then, we could state the extension of Theorem 3.

Theorem 5.2.1 (An Extension of the $\mathbb{Z}_r$ Tucker Lemma) *Let T be a triangulation of the octahedral subdivision of $S^{n,r}$, with vertex set V. Given any labelling $\lambda : V \to \mathbb{Z}_r \times \{1, \ldots, m\}$ such that $\lambda(\omega^i v) = \omega^i \lambda(v)$ for all $v \in V$ and no simplex is fully signed,*

$$\sum_{\tau \in S^{n,r}} \sum_{k \in K_n} \beta(\tau, k) \equiv 1 \pmod 2$$

Note that the maximum number of labels on a simplex is given by $m(r-1)$, so for the summation to be non-void, we must have $m(r-1) \geq n+1$, giving the contrapositive of Tucker's Lemma on a branched sphere.

Chapter 6

$\mathbb{Z}_r$ Borsuk-Ulam Theorem

Since Tucker's Lemma has such a natural extension to the Borsuk-Ulam Theorem, it is natural to ask if the $\mathbb{Z}_r$ version can be extended to a $\mathbb{Z}_r$ version of the Borsuk-Ulam theorem. In this section we will develop such an extension.

6.1 The $\mathbb{Z}_r$ Borsuk-Ulam Theorem

Theorem 4 (The $\mathbb{Z}_r$ Borsuk-Ulam Theorem) *For an even r, given any continuous function $f : S^{n(r-1),r} \to ((\mathbb{Z}_r \times \mathbb{R}^+) \cup \{0\})^{n(r-1)}$, there exists some $z \in S^{n(r-1),r}$ such that*

$$\sum_{j=0}^{r-1} \omega^{-j} f(\omega^j z) = 0.$$

As before, this has an equivalent statement:

Theorem 6.1.1 *For an even r, given any continuous function $f : S^{n(r-1),r} \to ((\mathbb{Z}_r \times \mathbb{R}^+) \cup \{0\})^{n(r-1)}$ such that $f(\omega^j z) = \omega^j f(z)$, there exists some $z \in S^{n(r-1),r}$ such that $f(z) = 0$.*

To show that this statement implies the Borsuk-Ulam theorem, we note that the function $\sum_{j=0}^{r-1} \omega^{-j} f(\omega^j z)$ satisfies the equivariant condition set forth. Therefore, there must be some point where the summation is zero, as required by the Borsuk-Ulam theorem.

On the other hand, the $\mathbb{Z}_r$ Borsuk-Ulam Theorem quickly implies this statement, since $0 = \sum_{j=0}^{r-1} \omega^{-j} f(\omega^j z) = \sum_{j=0}^{r-1} f(z)$ implies that $f(z) = 0$.

Finally, there is a continuous retraction version of the theorem:

Theorem 6.1.2 *There is no continuous retraction of the branched sphere $S^{n(r-1),r}$ to the branched sphere $S^{n(r-1)-1,r}$ that preserves the action ω.*

If we had such a retraction, by including the range space into $((\mathbb{Z}_r \times \mathbb{R}^+) \cup \{0\})^{n(r-1)}$, we would have a continuous function satisfying Theorem 6.1.1 that did not have a zero. On the other hand, if we had a continuous function satisfying Theorem 6.1.1 that did not have a zero, by normalizing the vectors in the range space we would have a function that continuously retracts $S^{n(r-1),r}$ to $S^{n(r-1)-1,r}$.

6.2 The Tucker $\mathbb{Z}_r$ Lemma and the $\mathbb{Z}_r$ Borsuk-Ulam Theorem

Just as we were able to find a way to use Tucker's Lemma to prove the Borsuk-Ulam Theorem constructively, it seems only natural that the analog should be true in the case of the branched sphere.

Theorem 6.2.1 *The $\mathbb{Z}_r$ Tucker's Lemma implies the $\mathbb{Z}_r$ Borsuk-Ulam Theorem.*

Proof: Given a continuous f from Theorem 6.1.1 and an $\epsilon > 0$, we triangulate $S^{n(r-1),r}$ with vertex set V so that any pair of neighboring vertices have images under f within $\epsilon\frac{\sin(2\pi/r)}{\sqrt{n}}$ units of each other.

For any vertex z of this triangulation, if $f = 0$, we are done. Otherwise, define a labelling $\lambda : V \to \mathbb{Z}_r \times \{1, 2, \ldots, n\}$ by first defining $j(z) = \text{argmax}_k |f_k(z)|$. Then we define $\lambda(z) = (\text{sgn}(f_{j(z)}(z)), \lceil j(z)/(r-1) \rceil)$. This defines a $\mathbb{Z}_r$ Tucker labelling of the branched sphere, as the coordinate j of the maximum does not change under an action by ω^j, but the sign at that coordinate does.

By $\mathbb{Z}_r$ Tucker's Lemma, we know that there must be some fully signed $r-1$-simplex. Since this simplex contains r elements, it must contain two elements with the same j value but different $\mathbb{Z}_r$ signs. Because these elements are within $\epsilon\frac{\sin(2\pi/r)}{\sqrt{n}}$ of each other, they must be within $\epsilon/\sqrt{n}$ of 0, as seen in Figure 6.1. Since these elements are the largest coordinates of f, the image under f of one of these vertices must be within ϵ of 0. Because ϵ was chosen arbitrarily, in the limit as $\epsilon \to 0$, there must be some z such that $f(z) = 0$. □

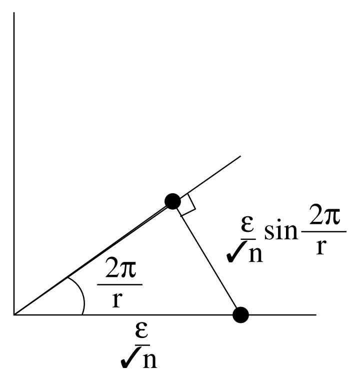

Figure 6.1: Two points within $\epsilon\frac{\sin(2\pi/r)}{\sqrt{n}}$ of each other must be within $\epsilon/\sqrt{n}$ of 0

Chapter 7

$\mathbb{Z}_r$ Fair Division

As before, we can use the previous $\mathbb{Z}_r$ results to prove fair division problems.

7.1 Our Problem

Suppose we now had a cake that we wish to divide into r equal shares. The equality of the shares will be assessed by a group of n people, who still have different ideas about what would be a fair division of the cake. We again make a series of parallel cuts along the cake that can be described by the points on the branched sphere. We will then find a division of the cake into r pieces using $n(r-1)$ cuts that will satisfy everyone within a preset amount ϵ.

This can be formalized as follows: Given a bounded set in $\mathbb{R}^k$ and a n different measures on the set, we want to find a series of cuts by $(k-1)$-dimensional parallel hyperplanes of the set into r portions such that each measure agrees that the division is fair. We will require that the measures be continuous (i.e. no point masses) and bounded.

We start by moving the cake so that it is tangent to the hyperplane $x_1 = 0$ and scaling it so that it is also tangent to the hyperplane $x_1 = 1$; we again use cuts by hyperplanes $x_1 = c$, where $0 < c < 1$. Now any point in the r-branched $n(r-1)$-sphere can be associated with a division by $n(r-1)$ knives and an assignment of the $n(r-1)+1$ pieces to portions in the following way: let $|z_i|$ = width of piece i and $\text{sgn}(z_i) = \omega^j$ according as the i^{th} piece is assigned to the j^{th} portion. Note that $\sum_{i=1}^{n(r-1)+1} |z_i| = 1$, as required by definition 5.1.2, the r-branched $n(r-1)$-dimensional sphere.

Given an $\epsilon > 0$, triangulate the branched sphere so that any two adjacent vertices have the measure of each portion within ϵ of each other, under every measure. For each vertex in the triangulation, we define a label of the vertex whose value is the number of the person measuring the largest absolute difference in the size of any two portions, and whose sign is the sign of the portion they feel is the largest. Because action by ω^j is equivalent to cycling the portions, the same person will measure the largest difference, but will have changed which piece is largest in accordance with the change by j. Therefore the labelling we have defined is a $\mathbb{Z}_r$ Tucker labelling.

By the $\mathbb{Z}_r$ Tucker Lemma, there must be a fully signed simplex. This would correspond to the same person measuring the largest absolute difference, but choosing every portion to be the largest. Since each vertex has a measure within ϵ of every other vertex, that person must measure each portion to be equivalent, within ϵ. Since this is the largest difference measured, everyone else must be in agreement that the portions differ by less the ϵ. As we let $\epsilon \to 0$, we have a sequence in the compact space $S^{n(r-1),r}$, which must have a convergent subsequence, whose limit point is 0. This point corresponds to everyone agreeing that the division is fair.

Appendix A

Appendix

A.1 Sperner's Lemma

We will now take a brief tangent to explain Sperner's Lemma. Sperner's Lemma is similar to Tucker's in that they both relate to labelings of triangulations and have powerful topological implications. While Tucker's Lemma is equivalent to the Borsuk-Ulam theorem (which we will show in Chapter 3), Sperner's Lemma is equivalent to the Brouwer Fixed Point theorem [7].

We start with an n-simplex and label the vertices of the simplex with the labels {1,...,n+1}. We then triangulate the simplex and label any vertex of the triangulation by one of the labels supporting that vertex. This is called a *Sperner labelling.* Within the Sperner labelling, we will say that a simplex if *fully labelled* if all of its labels are different — i.e. each of the labels $\{1, \ldots, n+1\}$ appears on one of the vertices of the simplex. We will be able to prove Sperner's Lemma by using Fan's version of Tucker's Lemma.

Theorem A.1.1 (Sperner's Lemma) *Given any Sperner labelled triangle, there exists an odd number of fully labelled simplices.*

Proof: Given a Sperner labelled simplex, introduce an additional $n+1$ vertices, with one corresponding to each $n-1$ dimensional face of the simplex, and connect this new vertex to every vertex in the face to which it corresponds. Label these vertices by the negation of the label of the vertex that lies opposite them — so that the vertex that is connected to the face supported by vertices 2 through $n+1$ is labelled -1, and the vertex that is connected to the face supported by vertices 1 and 3 through $n+1$ is labelled -2, etc. An example for the case $n = 2$ is shown in Figure A.1.

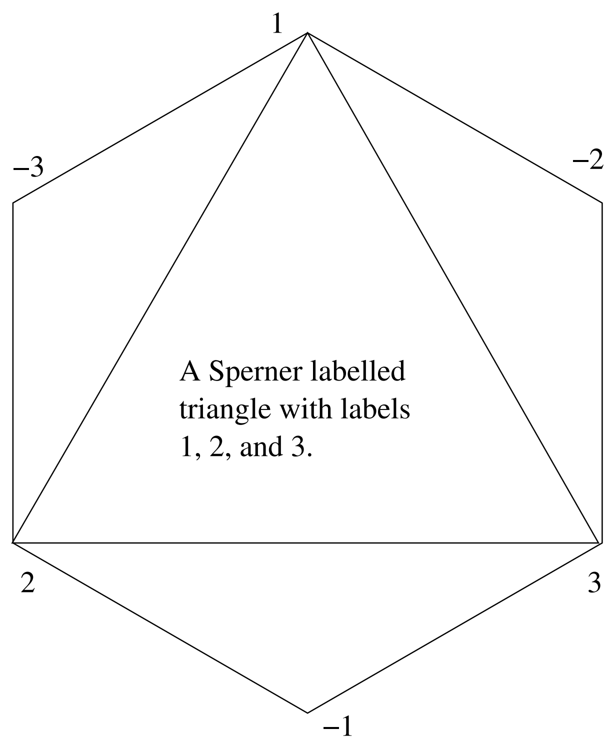

Figure A.1: Showing that Tucker's lemma implies Sperner's by direct construction.

As there are only $2(n+1)$ vertices defining the boundary of this ball, it is clear that opposite points have opposite labels. Furthermore, since the negative of any label occurs only once, it is clear that it can only be adjacent to labels that are positive and not oppositely labelled. Therefore, we can apply Corollary 2.4.3 and deduce that $\beta(1, 2, \ldots, n+1) + \beta(-1, -2, \ldots, -(n+1))$ is odd.

However, this ball was constructed so that no negative vertices are adjacent to each other. Therefore, $\beta(-1, -2, \ldots, -(n+1)) = 0$ and we may conclude that $\beta(1, 2, \ldots, n+1)$ is odd. Since the positive labels occur only in the original triangle, we may conclude that there are an odd number of simplices within the original triangle with all $n+1$ labels, in accordance with Sperner's Lemma. □

A.2 Homology of a Branched Sphere

Since we have defined this new topological object, it would be interesting to find the homology of the branched sphere. Homology is used to study the underlying topology of an object, and can tell how that object is related to other objects. One may think of the homology as the result of embedding n-dimensional spheres into the object.

Since the branched ball $B^{n,r}$ is a cone, it must be acyclic and have trivial homology groups in every dimension. Since the zero dimensional branched sphere $S^{0,r}$ is a collection of r points, we may calculate its homology (with integral coefficients) to find that $H_0(S^{0,r}) \cong \mathbb{Z}^r$, or in reduced homology $\widetilde{H}_0(S^{0,r}) \cong \mathbb{Z}^{r-1}$. All other homologies must of course be trivial, since the sphere has dimension zero.

For higher dimension branched spheres, we will build up from lower dimensions by examining each sphere $S^{n,r}$ as the join of $n+1$ r-element sets. In order to do this, we denote by $[m]$ the set of m elements.

Theorem A.2.1 (Homology of a Join) *Let $\mathcal{S}$ denote the simplicial complex $\star_{i=1}^{n+1}[m_i]$. Then*

$$\widetilde{H}_n(\mathcal{S}) \cong \mathbb{Z}^{\prod_{i=1}^{n+1}(m_i-1)},$$

and all other homologies are trivial.

Proof: We will proceed by inducting on n and m_n. When $n=0$, then $\mathcal{S}=[m_1]$, so $\widetilde{H}_0(\mathcal{S}) \cong \mathbb{Z}^{m_1-1}$. Additionally, if any $m_i=1$, then $\mathcal{S}$ is a cone over the remainder of the join, and $\mathcal{S}$ has trivial homology in all dimensions, as expected. Finally, if any $m_i=2$, then $\mathcal{S}$ is a suspension over the remainder of the join, and the homology of $\mathcal{S}$ is isomorphic to the homology of the join prior to the suspension, although in one less dimension, again as expected.

For our induction, we will continue to denote by $\mathcal{S}$ the complex $\star_{i=1}^{n+1}[m_i]$, where $m_{n+1}>2$. We will also denote by $\mathcal{T}$ the complex $\star_{i=1}^{n}[m_i] \star [m_{n+1}-1]$, by $\mathcal{T}'$ the complex $\star_{i=1}^{n}[m_i] \star \{m_{n+1}\}$, and by $\mathcal{S}'$ the complex $\star_{i=1}^{n}[m_i]$. In this manner, the complex $\mathcal{S}$ is the union of the two complexes $\mathcal{T}$ and $\mathcal{T}'$, whose intersection is $\mathcal{S}'$. By our induction hypothesis, $\widetilde{H}_n(\mathcal{T}) \cong \mathbb{Z}^{\prod_{i=1}^{n}(m_i-1)(m_{n+1}-2)}$, $\widetilde{H}_n(\mathcal{S}') \cong \mathbb{Z}^{\prod_{i=1}^{n}(m_i-1)}$, and all other homologies are trivial.

Since $T \cup T' = S$ and $T \cap T' = S'$, we have the long exact sequence in reduced homology:

$$\cdots \widetilde{H}_j(\mathcal{S}') \to \widetilde{H}_j(\mathcal{T}) \oplus \widetilde{H}_j(\mathcal{T}') \to \widetilde{H}_j(\mathcal{S}) \to \widetilde{H}_{j-1}(\mathcal{S}') \cdots ,$$

or, since most of these homologies are trivial, we have the short exact sequence

$$0 \to \widetilde{H}_n(\mathcal{T}) \xrightarrow{\phi} \widetilde{H}_n(\mathcal{S}) \xrightarrow{\psi} \widetilde{H}_{n-1}(\mathcal{S}') \to 0,$$

in which the map ϕ is inclusion and the map ψ is induced by taking a cycle in $\mathcal{S}$, taking the boundary of the portion lying in $\mathcal{T}$, and including the results in $\mathcal{S}'$. By defining another map $r : \widetilde{H}_n(\mathcal{S}) \to \widetilde{H}_n(\mathcal{T})$ that is a restriction map, we have a map such that $r \circ \phi = i_{\widetilde{H}_n(\mathcal{T})}$. By Munkres [10] Theorem 23.1, we know that the sequence splits, so that

$$\begin{aligned} \widetilde{H}_n(\mathcal{S}) &\cong \phi(\widetilde{H}_n(\mathcal{T})) \oplus \widetilde{H}_{n-1}(\mathcal{S}') \\ &\cong \widetilde{H}_n(\mathcal{T}) \oplus \widetilde{H}_{n-1}(\mathcal{S}') \end{aligned}$$

Therefore,

$$\begin{aligned} \widetilde{H}_n(\mathcal{S}) &\cong \mathbb{Z}^{\prod_{i=1}^{n}(m_i-1)(m_{n+1}-2)} \oplus \mathbb{Z}^{\prod_{i=1}^{n}(m_i-1)} \\ &\cong \mathbb{Z}^{\prod_{i=1}^{n}(m_i-1)(m_{n+1}-2)+\prod_{i=1}^{n}(m_i-1)} \\ &= \mathbb{Z}^{\prod_{i=1}^{n}(m_i-1)(m_{n+1}-1)} \\ &= \mathbb{Z}^{\prod_{i=1}^{n+1}(m_i-1)} \end{aligned}$$

as desired. $\square$

Corollary A.2.2 (Homology of a Branched Sphere) *The homology of a branched sphere is given by*

$$\widetilde{H}_n(S^{n,r}) \cong \mathbb{Z}^{(r-1)^{n+1}}.$$

All other homologies are trivial.

Proof: We have already noted that $S^{n,r} \cong \star_{i=1}^{n+1}\{\{i\} \times \mathbb{Z}_r\}$, the $n+1$-fold join of r element sets. By the previous theorem,

$$\widetilde{H}_n(S^{n,r}) \cong \mathbb{Z}^{\prod_{i=1}^{n+1}(r-1)} = \mathbb{Z}^{(r-1)^{n+1}}$$

and all other homologies are trivial. □

One may view the generators of the homology groups by first fixing a point from each of the r element sets and then choosing another point from each of the sets.

A.3 Future Work

There are many things I would like to do:

- $\mathbb{Z}_r$ **Tucker Lemma.** Prove the extension of Fan's theorem for general r.

- **Ham Sandwich Theorem.** I would like to show how it can be extended to the Branched Sphere.

- **Sperner's Lemma** I have already discussed how Fan's version of Tucker's Lemma implies Sperner's Lemma. It would be nice to show how Tucker's Lemma on its own implies Sperner's, as we know it must since Tucker ⇒ Borsuk-Ulam ⇒ Brouwer ⇒ Sperner. Maybe we could also see what happens if we are using a branched ball.

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